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A CHARACTERIZATION OF TEICHMÜLLER DIFFERENTIALS

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1. Introduction

1.1. A quasiconformal affine mapping Tz has the expression

$$Tz = A(z + ke^{2i\theta}\overline{z}) = Ae^{i\theta}([e^{-i\theta}z] + k\overline{[e^{-i\theta}z]}),$$

where $0 < k < 1$ if T is not conformal. From this expression we conclude that

(1) there is a unique rectangular coordinate system whose image is rectangular, and

(2) in terms of this pair of systems, T can be expressed as a pure stretch: writing $w = u + iv$, depending on the scaling, and $K = (1+k)/(1-k)$, we have

$$Tw = \begin{cases} Ku + iv & \text{(height preserving)}, \\ \sqrt{K}u + iv/\sqrt{K} & \text{(area preserving)}. \end{cases}$$

Let $g: R \to S$ be an orientation preserving homeomorphism between the compact or finitely punctured compact Riemann surfaces R and S of hyperbolic type. Starting with the assumption that a condition analogous to property (1) above holds, we will establish that the analogue to property (2) follows; that is, we will establish by an explicit construction that there is a Teichmüller mapping in the homotopy class of g. In §9 we will show that the unique axis theorem for hyperbolic (pseudo-Anosov) elements of the Teichmüller modular group is a consequence.

In the last part of the paper, it will be shown that the analogue to property (1) in fact holds. This leads to a geometric proof of the Teichmüller mapping theorem (§10).

Our first main theorem is related, via the theory of measured foliations, to the following result of Masur [11] as completed in Gardiner-Masur [3]. Namely, an equivalence class of transverse measured foliations on a surface determines a unique complex structure in terms of which it is realized

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as the horizontal and vertical foliations of a unique quadratic differential. There is a dictionary between measured foliations and quadratic differentials on Riemann surfaces of finite types. Our work is carried out in the latter category, which we believe has a number of advantages.

1.2. We arrive at a necessary condition as follows. A holomorphic quadratic differential φ on a Riemann surface R has, locally and away from its zeros, an integral function of its square root,

$$w = u + iv = \Phi(z) = \int^{z} \sqrt{\varphi(z)}\, dz$$

(see, e.g., [12]). Two different local function elements are connected by the equation

$$\Phi_2(z) = \pm\Phi_1(z) + \text{const}.$$

We have a well-defined length element $|dw| = |\varphi(z)|^{1/2}|dz|$, which is nothing but the Euclidean length element in the Φ-plane. The norm

$$\|\varphi\| = \iint_R |\varphi|\, dx\, dy$$

is the surface area. This metric determines a singular Euclidean geometry on R. There exist the following elements $|dv|$ of "height" and $|du|$ of "horizontal length":

$$|dv| = |\operatorname{Im}(\sqrt{\varphi(z)}\, dz)|,$$
$$|du| = |\operatorname{Re}(\sqrt{\varphi(z)}\, dz)| = |\operatorname{Im}(\sqrt{-\varphi(z)}\, dz)|.$$

It is evidently enough to consider $|dv|$.

For any piecewise smooth closed loop γ on R which is not contractible to a point we define the "height" of its free homotopy class $[\gamma]$ by

$$h_\varphi(\gamma) := \inf_{\tilde\gamma \sim \gamma} \int_{\tilde\gamma} |dv| > 0 \qquad (\varphi \not\equiv 0),$$

where $\tilde\gamma$ runs over $[\gamma]$. The "mapping-by-heights" g_h associates with each differential φ on R a differential ψ on S which is uniquely determined by the property that

$$h_\psi(g(\gamma)) = h_\varphi(\gamma) \quad \forall \gamma \text{ on } R$$

(for existence and uniqueness of $\psi = g_h(\varphi)$ see [5]).

From the assumption that there exists a Teichmüller mapping f in the homotopy class of g, we can easily derive a necessary condition for the associated quadratic differentials φ on R and ψ on S, which we call the Teichmüller differentials associated with g. Locally and away from the zeros of φ, f has the representation

$$f = \Psi^{-1} \circ F \circ \Phi,$$

with $\Phi = \int \sqrt{\varphi}$, $\Psi = \int \sqrt{\psi}$, and F the horizontal stretching by the dilatation K of f,

$$F : w = u + iv \to F(w) = Ku + iv.$$

It follows for every arc γ and its image $f(\gamma)$ that

$$\int_\gamma |\operatorname{Im}(\sqrt{\varphi(z)}\,dz)| = \int_{f(\gamma)} |\operatorname{Im}(\sqrt{\psi(z)}\,dz)|,$$

just by locally passing to the Φ- and Ψ-plane respectively. Thus, ψ is the image of φ under the mapping g_h.

On the other hand, "horizontal lengths" are stretched by K, since

$$\int_{f(\gamma)} |\operatorname{Re}(\sqrt{\psi(z)}\,dz)| = K \int_\gamma |\operatorname{Re}(\sqrt{\varphi(z)}\,dz)|.$$

Looking at $-\varphi$ and $-\psi$ instead of φ and ψ, we are back to "heights," namely

$$Kh_{-\varphi}(\gamma) = h_{-\psi}(f(\gamma)).$$

1.3. Denote the vector space of holomorphic differentials of finite norm by $Q(R)$ and the unit sphere in $Q(R)$ by $Q_0(R)$. The homeomorphism $g: R \to S$ determines the homeomorphism $g_h: Q(R) \to Q(S)$ and also the "normed mapping by heights", namely the homeomorphism $g_\#: Q_0(R) \to Q_0(S)$ between the unit spheres defined by

$$g_\#(\varphi) = \frac{g_h(\varphi)}{\|g_h(\varphi)\|} \quad \forall \varphi \in Q_0(R).$$

The differential $g_\#(\varphi)$ is uniquely determined by the following two properties:

(i) $\|g_\#(\varphi)\| = 1$, and

(ii) $g_\#(\varphi)$ has heights proportional to those of φ on corresponding loops.

For, let $\psi = g_\#(\varphi)$ and $\tilde{\psi}$ be two such differentials, with proportionality factors λ and $\tilde{\lambda}$, respectively. Then ψ/λ^2 and $\tilde{\psi}/\tilde{\lambda}^2$ have the same heights as φ on corresponding loops. Therefore, by the uniqueness theorem [5] we have $\tilde{\psi}/\tilde{\lambda}^2 = \psi/\lambda^2$, hence $\lambda = \tilde{\lambda}$ and $\psi = \tilde{\psi}$.

With this normalization of the Teichmüller differentials φ and $\psi = g_\#(\varphi)$, the affine mapping F becomes

$$F(w) = \sqrt{K}u + i\frac{1}{\sqrt{K}}v.$$

The area with respect to the two metrics induced by φ and ψ is preserved, and the total area, which is equal to the norm, is one.

The normed mapping-by-heights $g_\#$ applied to the Teichmüller differential φ satisfies $g_\#(\varphi) = \psi$ and $g_\#(-\varphi) = -\psi$, or in short,

$$g_\#(-\varphi) = -g_\#(\varphi).$$

This is the necessary condition we were aiming at. The goal of the first part of this paper is to establish its sufficiency. We will then have:

Theorem 1.3. *A necessary and sufficient condition for $\varphi \in Q_0(R)$ and hence $g_\#(\varphi) \in Q_0(S)$ to be associated with a Teichmüller mapping $f: R \to S$ homotopic to g is that*

$$(\#) \qquad g_\#(-\varphi) = -g_\#(\varphi).$$

Remark. Teichmüller uniqueness theorem implies that the solution φ is unique.

1.4. The proof of Theorem 1.3 needs a deeper study of the mapping g_h (we will usually first work with g_h and then pass to $g_\#$, if necessary). The introduction of "height of arc between trajectories" instead of just height of closed loops and the proof of its invariance under g_h enable us to show that:

g determines a bijection between the regular trajectories of any φ and those of $g_h(\varphi)$ and preserves the vertical distances between them.

This is the second main result of the paper and in fact allows the construction of the Teichmüller mapping if the functional equation is fulfilled.

The discussion will be carried out first for compact surfaces. In §8 the generalization to compact surfaces with punctures will be made. There is a version of Theorem 1.3 for compact bordered surfaces as well, but we will not carry out the details here.

2. Review of previous results

2.1. A trajectory α of $\varphi \in Q(R)$ is a maximal arc along which $\varphi(z)\,dz^2 > 0$ for tangential dz. It is either closed or a simple arc. In the latter case, it is called critical if it tends, in at least one direction, to a critical point of φ (i.e., a zero); otherwise it is called noncritical or regular. A vertical trajectory of φ is a trajectory of $-\varphi$. A geodesic of φ is composed of φ-straight arcs, i.e., arcs along which $\arg\{\varphi(z)\,dz^2\} = \text{const}$. These meet at zeros of φ, where both angles are at least $2\pi/(m+2)$, m the order of the zero. We call it a horizontal geodesic if all its sides are horizontal (i.e., $\arg\{\varphi(z)\,dz^2\} = 0$).

An oriented, complete arc is called a left extreme horizontal geodesic if in addition the angles along its left-hand side at its vertices are equal to $2\pi/(m+2)$. Right extreme horizontal geodesics are defined similarly.

The critical graph of a differential φ is the union of its zeros and the horizontal rays emanating from them. If the critical graph contains no closed loops, then φ is called *admissible.* We will however not use this condition in our proofs.

2.2. Most of our work will be carried out in $\mathbb{D}$, the universal covering surface of R, which we may take to be the unit disk. Denote the unit circle by $\partial\mathbb{D}$. Each $\varphi \in Q(R)$ has a lift $\tilde{\varphi}$ to $\mathbb{D}$, uniquely determined by the projection map. Generally we denote a lifted object by a tilde; thus $\tilde{\alpha}$ is a lift of the trajectory α of φ, which is of course a trajectory of $\tilde{\varphi}$.

As shown in [6], a geodesic $\tilde{\alpha}$ of $\tilde{\varphi}$ has two distinct endpoints on $\partial\mathbb{D}$. Distinct horizontal geodesics $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ which have their endpoints in common project to parallel simple loops α_1 and α_2 in R, bounding an annulus swept out by closed horizontal trajectories.

2.3. Assume now R is compact. Then it was shown in [9] that distinct trajectory rays from z_0, $z_1 \in \mathbb{D}$ cannot have a common endpoint unless they project to parallel closed loops.

An important fact for a compact surface is that the metric in $\mathbb{D}$ determined by a holomorphic quadratic differential $\tilde{\varphi} \not\equiv 0$ is complete, which means that the boundary $\partial\mathbb{D}$ is at infinite distance from any interior point (see, e.g., [7]).

The following result about convergence of trajectories will be used in §5. Suppose $\{\varphi_n\}$ is a sequence of normalized, holomorphic quadratic differentials which converges locally uniformly to φ. Let $\tilde{\varphi}_n$ and $\tilde{\varphi}$ be the lifted differentials to $\mathbb{D}$. Let $\tilde{\alpha}$ be a regular trajectory ray of $\tilde{\varphi}$ with initial point $z_0 \in \mathbb{D}$ and assume $z_n \to z_0$. Then any sequence of geodesic rays $\tilde{\alpha}_n$ of $\tilde{\varphi}_n$ with initial points z_n and leaving z_n with the limiting direction of $\tilde{\alpha}$ tends uniformly to $\tilde{\alpha}$ (see [7, Theorem 2]).

2.4. Suppose that $g\colon R \to S$ is an orientation preserving homeomorphism of R onto another Riemann surface S. It lifts to a homeomorphism $\tilde{g}\colon \mathbb{D} \to \mathbb{D}$ which in turn extends to homeomorphism $\tilde{g}\colon \partial\mathbb{D} \to \partial\mathbb{D}$. The map g also induces an isomorphism of the fuchsian covering groups $\theta\colon G \to H$ by the formula

$$\tilde{g}(Tz) = \theta(T)\tilde{g}(z), \quad \forall T \in G, \forall z \in \mathbb{D} \cup \partial\mathbb{D}.$$

Conversely, every isomorphism $\theta\colon G \to H$ is induced by a homeomorphism $g\colon R \to S$ whose homotopy class is uniquely determined by θ. The isomorphisms θ and θ_1 determined by different lifts of g are related as

$$\theta_1(T) = B\theta(ATA^{-1})B^{-1}$$

for some $A \in G$ and $B \in H$, $\forall T \in G$.

2.5. The fundamental tool in our work is the realtionship of the maps $\tilde{g}: \partial\mathbb{D} \to \partial\mathbb{D}$ and $g_\#$.

Assume first that $\varphi \in Q_0(R)$ is admissible. Then each component of the critical graph of $\tilde{\varphi}$ in $\mathbb{D}$ contains only a finite number of critical points. We showed in [7, Theorem 3] that two points $x_1, x_2 \in \partial\mathbb{D}$ are the endpoints of a horizontal trajectory (resp., extreme horizontal geodesic) of the lift $\tilde{\varphi}$ in $\mathbb{D}$ if and only if $\tilde{g}(x_1)$ and $\tilde{g}(x_2)$ are the endpoints of a horizontal trajectory (resp., extreme horizontal geodesic) of $g_\#(\varphi)^\sim$. Moreover, $\tilde{g}$ and $g_\#$ determine a certain correspondence between critical points of φ and $g_\#(\varphi)$. The simplest situation is when neither φ nor $g_\#(\varphi)$ have saddle connections (critical horizontal trajectories of finite length connecting critical points). In this case, there is a one-to-one correspondence between critical points of φ and $g_\#(\varphi)$, and more strongly, critical points of the lifted $\tilde{\varphi}$ and $g_\#(\varphi)^\sim$. The correspondence is given as follows. Corresponding to each horizontal ray $\tilde{\alpha}$ of $\tilde{\varphi}$ emanating from a critical point $\zeta \in \mathbb{D}$ with endpoint $x \in \partial\mathbb{D}$ is a horizontal ray $g_\#(\tilde{\alpha})$ of $g_\#(\varphi)^\sim$ emanating from a critical point $g_\#(\zeta)$ and terminating at $\tilde{g}(x) \in \partial\mathbb{D}$.

The introduction of vertical distance of trajectories will enable us to show that the correspondence of regular horizontal trajectories by g_h is generally true, i.e., without the assumption of admissibility.

2.6. For the proof of the convergence of geodesic connections of boundary points of $\mathbb{D}$ we will need a uniform version of Lemma 3 in [7], namely:

Lemma 2.6. *Let $\varphi_n \to \varphi$ and $\|\varphi_n\| = \|\varphi\| = 1$. Let $\{z_n\}$ and $\{z'_n\}$ be two sequences of points in $\mathbb{D}$ with $z_n \to \zeta$ and $z'_n \to \zeta'$, $\zeta \neq \zeta' \in \partial\mathbb{D}$. Then there exists $r < 1$ such that the $\tilde{\varphi}_n$-geodesic $\tilde{\tau}_n$ between z_n and z'_n has a point in $\mathbb{D}_r$: $|z| < r$ for all n.*

The proof rests on the fact that the closed geodesics γ_n of φ_n in a fixed free homotopy class of R tend to the closed geodesic γ of φ. (If γ is not unique, the sequence $\{\varphi_n\}$ has a subsequence and a choice of γ for which this is true.) With this supplementary remark, the proof of Lemma 3 in [7] can be applied.

2.7. The same is true for Theorem 1 of [7]. The uniform version which we need is

Lemma 2.7. *Let $\varphi_n \to \varphi$ and $\|\varphi_n\| = \|\varphi\| = 1$. Let $\{\zeta_n\}$ and $\{\zeta'_n\}$ be two sequences of boundary points of $\mathbb{D}$ which tend to ζ and $\zeta' \neq \zeta$*

respectively. Then the $\tilde{\varphi}$-geodesics $\tilde{\tau}_n$ between ζ_n and ζ'_n tend to the $\tilde{\varphi}$-geodesic $\tilde{\tau}$ between ζ and ζ' uniformly in the Euclidean metric of the disk. If the latter is not unique, there is a subsequence and a choice of $\tilde{\tau}$ for which this is true.

The proof is a straightforward adaptation of (ii) of Theorem 1 in [7].

3. The vertical distance of horizontal geodesics

3.1. Let $\varphi \not\equiv 0$ be a holomorphic quadratic differential on a compact Riemann surface R of genus ≥ 2. Let α_1 and α_2 be two not necessarily distinct given trajectories or, more generally, horizontal geodesics of φ. An arc γ joining a point $P_1 \in \alpha_1$ to a point $P_2 \in \alpha_2$ is said to be freely homotopic to an arc γ', joining $P'_1 \in \alpha_1$ to $P'_2 \in \alpha_2$, if there exists a continuous deformation of γ into γ' which lets slide the points P_i on α_i, respectively. Such a family of freely homotopic arcs is called a free family of arcs and denoted by $[\gamma]$ rel (α_1, α_2).

Definition 3.1. Given $\varphi \in Q(R)$, the height of the free family $[\gamma]$ rel (α_1, α_2) of piecewise smooth arcs or, alternatively, the vertical distance between the trajectories (or horizontal geodesics) α_1 and α_2 in the homotopy class of γ is

$$\Delta_\gamma(\alpha_1, \alpha_2) := \inf_{\gamma' \sim \gamma} \int_{\gamma'} |dv|,$$

where γ' varies in the free homotopy class of γ.

Remarks. (1) Since $dv = 0$ along horizontal arcs, we can add arbitrary subintervals of α_1 and α_2 respectively at the ends of an arc γ. We therefore get the same number if we fix the points $P_1 \in \alpha_1$ and $P_2 \in \alpha_2$ and just work with the usual homotopy of arcs with fixed endpoints.

(2) Let $\tilde{\varphi}$ be the lift of φ to the universal covering surface $\mathbb{D}$. Choose an arbitrary lift z_1 of P_1. Then the lift $\tilde{\gamma}$ of γ with initial point z_1 determines a lift z_2 of P_2. Let $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ be the lifts of α_1 and α_2 through z_1 and z_2, respectively. The vertical distance of α_1 and α_2 in the homotopy class of γ becomes

$$\Delta_\gamma(\alpha_2, \alpha_2) = \Delta(\tilde{\alpha}_1, \tilde{\alpha}_2) = \inf_{\{\tilde{\gamma}\}} \int_{\tilde{\gamma}} |\mathrm{Im}(\sqrt{\tilde{\varphi}(z)}\, dz)|,$$

where the infimum is taken over all arcs $\tilde{\gamma}$ joining z_1 to z_2 (or joining any point of $\tilde{\alpha}_1$ to a point of $\tilde{\alpha}_2$). This means that in the universal covering surface we can just speak of the vertical distance of two trajectories or horizontal geodesics.

3.2. The infimum is in fact a minimum, and there are many arcs along which it is assumed. A special one is the shortest connection $\tilde{\gamma}_0$ (with respect to the $\tilde{\varphi}$-metric) of z_1 and z_2. Since the boundary $\partial\mathbb{D}$: $|z| = 1$ is at infinite $\tilde{\varphi}$-distance from any point $z \in \mathbb{D}$ (see, e.g., [7, Lemma 2]), there exists such a curve $\tilde{\gamma}_0$ (see, e.g., [12, p. 84]). It is uniquely determined by its two endpoints.

Lemma 3.2. *The vertical distance of two trajectories or, more generally, horizontal geodesics* $\tilde{\alpha}_1$ *and* $\tilde{\alpha}_2$ *of* $\tilde{\varphi}$ *is equal to*

$$\Delta(\tilde{\alpha}_1, \tilde{\alpha}_2) = \int_{\tilde{\gamma}_0} |d\tilde{v}|,$$

where $\tilde{\gamma}_0$ *is the shortest connection of two arbitrarily fixed points* $z_1 \in \tilde{\alpha}_1$ *and* $z_2 \in \tilde{\alpha}_2$.

Proof. Let $\tilde{\gamma}$ be an arbitrary (piecewise smooth) arc joining z_1 and z_2 (Figure 1). It is, together with $\tilde{\gamma}_0$, contained in a disk $\mathbb{D}_r$: $|z| < r < 1$. There are only finitely many zeros of $\tilde{\varphi}$ in $\mathbb{D}_r$. Look at the relatively regular trajectories through $\tilde{\gamma}_0$, i.e., the horizontal intervals cutting $\tilde{\gamma}_0$ but not meeting a zero in $\mathbb{D}_r$. We continue these, from $\tilde{\gamma}_0$, in both directions, up to their first intersection with $\partial\mathbb{D}_r$. The configuration we get consists of a finite number of open horizontal strips. Each of these is traversed exactly once by $\tilde{\gamma}_0$: since $\tilde{\gamma}_0$ is a geodesic, it cannot have two points in common with a trajectory unless it contains it, which is impossible.

Let b_i be the heights of the strips traversed by $\tilde{\gamma}_0$. Then

$$\int_{\tilde{\gamma}_0} |d\tilde{v}| = \sum_i b_i.$$

On the other hand, each of these strips is also traversed by γ, whence

$$\int_{\tilde{\gamma}} |d\tilde{v}| \geq \sum_i b_i,$$

proving that the height of $\tilde{\gamma}_0$ is minimal.

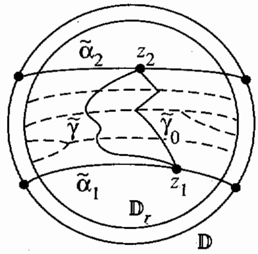

FIGURE 1

4. Convergence of vertical distances

4.1. Let $\{\varphi_n\}$ be a sequence of holomorphic quadratic differentials on R converging locally uniformly to $\varphi \not\equiv 0$. Let α_1 and α_2 be two regular horizontal trajectories of φ and let $P_1 \in \alpha_1$ and $P_2 \in \alpha_2$ be given points. Choose two sequences of points $P_{1n} \to P_1$ and $P_{2n} \to P_2$ such that the φ_n trajectories α_{1n} through P_{1n} and α_{2n} through P_{2n} are noncritical.

Let γ be a φ-geodesic from P_1 to P_2. In small neighborhoods N_1 of P_1 and N_2 of P_2 choose smooth arcs τ_{1n} and τ_{2n} connecting P_{1n} to P_1 and P_{2n} to P_2, respectively. There is a φ_n-geodesic γ_n from P_{1n} to P_{2n} uniquely determined by the requirement that γ'_n be homotopic to γ, where γ'_n is formed from γ_n by adjoining τ_{1n} and τ_{2n}.

Lemma 4.1.

$$\lim \int_{\gamma_n} |dv_n| = \int_{\gamma} |dv|.$$

That is, the φ_n-distance $\Delta_{\gamma_n}(\alpha_{1n}, \alpha_{2n})$ converges to the φ-distance $\Delta_\gamma(\alpha_1, \alpha_2)$, where the class $[\gamma_n]$ with fixed endpoints P_{1n} and P_{2n} is related to the class $[\gamma]$ with fixed endpoints P_1 and P_2 as described above.

Proof. The proof is performed in the universal covering surface $\mathbb{D}$. Choose lifts $\tilde{\alpha}_1$ of α_1 and $\tilde{\gamma}$ of γ from a point z_1 over P_1. The endpoint z_2 of $\tilde{\gamma}$ over P_2 lies on a uniquely determined lift $\tilde{\alpha}_2$ of α_2. Lifting the small neighborhoods N_1 and N_2 we can uniquely determine lifts z_{1n} of P_{1n} and z_{2n} of P_{2n} which converge to z_1 and z_2, respectively. Likewise, the lift $\tilde{\gamma}_n$ of γ_n from z_{1n} terminates at z_{2n}. Because of the uniqueness of geodesics the φ_n-geodesics $\tilde{\gamma}_n$ converge to $\tilde{\gamma}$. Hence, by the locally uniform convergence, $\lim \int_{\tilde{\gamma}_n} |d\tilde{v}_n| = \int_{\tilde{\gamma}} |d\tilde{v}|$. q.e.d.

We can proceed similarly with right and left extreme horizontal geodesics (see [7, p. 371]). Any such geodesic of φ is approximated by regular horizontal trajectories of φ, either from the right or from the left.

Finally, the same is true if the limit of the horizontal trajectories α_{in} is any horizontal geodesic α_i of φ.

5. The mapping-by-heights for simple differentials

5.1. For simple differentials, the correspondence of trajectories determined by g_h is a matter of definition.

Recall that given a simple loop γ, the simple differential $\varphi[\gamma]\,dz^2$ corresponding to γ is uniquely characterized up to a positive constant by the fact that all its regular trajectories are closed and lie in the free homotopy class $[\gamma]$ of γ. That is, the complement of the critical graph in R is a cylinder whose separating curves lie in $[\gamma]$. The simple differentials are dense in $Q(R)$ ([10], [5]).

Let φ be a simple differential on R, with b the height of its cylinder. Choose any closed trajectory α. Denote by $\psi = g_h(\varphi)$ the simple differential on S with closed trajectories β freely homotopic to $g(\alpha)$ and with the same height of cylinder b. That φ and ψ have the same heights on corresponding closed loops is a consequence of the invariance of the geometric intersection numbers (see [5]).

We now have

Lemma 5.1. *Let R and S be compact Riemann surfaces of genus ≥ 2 and let $g: R \to S$ be an orientation preserving homeomorphism. Then, for any simple differential φ on R, g_h takes the regular (i.e., closed) trajectories of φ into those of $\psi = g_h(\varphi)$.*

Proof. Let A be the cylinder of φ with an arbitrarily chosen orientation. Then g induces an orientation of the cylinder B of ψ. To any closed trajectory α of φ we assign the closed trajectory β of ψ which subdivides B in the same way as α subdivides A. We set $\beta = g_h(\alpha)$.

5.2. In $\mathbb{D}$ the correspondence is less trivial. Fix a lift of g and denote by $\tilde{g}: \partial\mathbb{D} \to \partial\mathbb{D}$ its extension to the boundary (one may think of g as being quasiconformal). A connected lift of the cylinder A of φ is swept out by the lifts $\tilde{\alpha}$ of the closed trajectories α with the same initial and endpoints p and q on $\partial\mathbb{D}$, respectively. We call this a slice. The points p and q are the fixed points of a cover transformation T of $\mathbb{D}$. The points $r = \tilde{g}(p)$ and $s = \tilde{g}(q)$ are the fixed points of the cover transformation $\theta(T)$ corresponding to T by the group isomorphism θ. The points r and s are the endpoints of a slice of $\tilde{\psi}$. The lifted map $\tilde{g}_h: \tilde{\varphi} \to \tilde{\psi}$ is well defined, since the lifts $\tilde{\varphi}$ and $\tilde{\psi}$ of φ and ψ, respectively, are well defined. But we can now also define $\tilde{\beta} = \tilde{g}_h(\tilde{\alpha})$ as the oriented trajectory $\tilde{\beta}$ connecting r with s and subdividing its slice in the same way as $\tilde{\alpha}$ subdivides its (oriented) slice. Orientation is gaken over by distinguishing initial and endpoints. We now have

Lemma 5.2. *Let $g_h: \alpha \to \beta$ be the mapping of trajectories defined in the preceding lemma. Fix a lift of g and the induced boundary homeomorphism $\tilde{g}: \partial\mathbb{D} \to \partial\mathbb{D}$. Then $\tilde{g}$ induces a well-defined mapping of the regular trajectories $\tilde{\alpha}$ of $\tilde{\varphi}$ onto the regular trajectories $\tilde{\beta}$ of $\tilde{\psi}$. Corresponding trajectories have corresponding initial and endpoints and subdivide their slices proportionally. We set $\tilde{\beta} = \tilde{g}_h(\tilde{\alpha})$.*

Note that this correspondence in general does not go over to the critical points and critical trajectories, as the following example shows (Figure 2).

We take two pentagons R and S with sides parallel to the axes. R is a bus shaped region, while S is a rectangle with an additional distinguished point or vertex on one of the vertical sides. The heights of the two figures are the same, and the vertices with number 3 are on the same horizontal (dotted) line. This line subdivides the figures into two rectangles R_1, R_2 and S_1, S_2, respectively. The quadratic differentials, in terms of the plane parameter, are $\varphi = \psi = 1$. The two differentials φ and ψ correspond to each other via the mapping by heights. This follows from the fact that R_1 and S_1 have the same height, and so do R_2 and S_2. The differential φ has a zero, namely the corner pointing to the inside, while the differential ψ has no zero (the zero and a first-order pole cancel at the vertex 3).

After conformal mapping onto the unit disk and completion by reflection we get Figure 3. We have two annuli, R_1', R_2' and S_1', S_2',

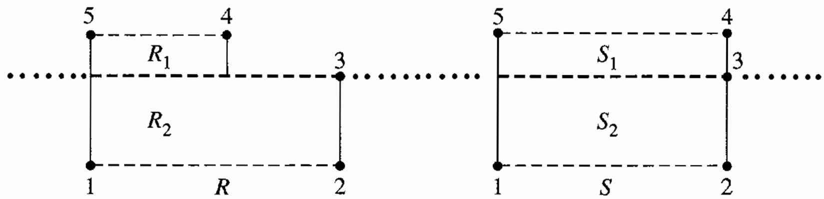

FIGURE 2

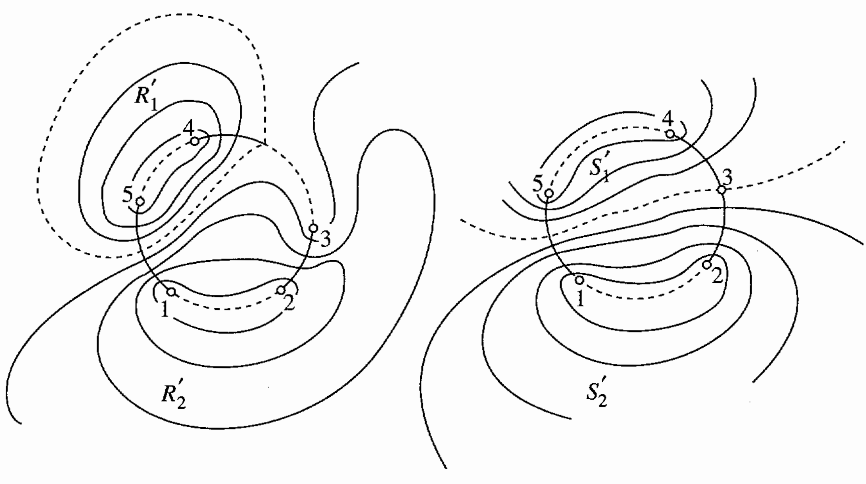

FIGURE 3

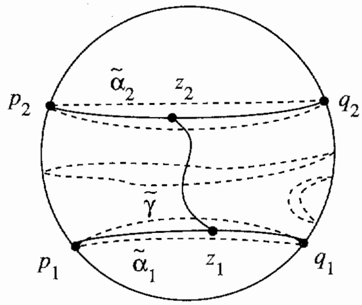

FIGURE 4

respectively, separated by a component of the horizontal critical graph (dotted lines in the figure).

5.3. To show that the mapping-by-heights g_h leaves vertical distances unchanged, let α_1 and α_2 be arbitrary closed trajectories of the simple differential φ. Let γ be an arc connecting a point $P_1 \in \alpha_1$ with a point $P_2 \in \alpha_2$. Choose a point $z_1 \in \mathbb{D}$ above P_1 and denote the lifts of α_1 and γ from the point z_1 by $\tilde{\alpha}_1$ and $\tilde{\gamma}$, respectively. The lift $\tilde{\alpha}_2$ of α_2 is determined by the endpoint z_2 of $\tilde{\gamma}$. Let p_1, q_1 and p_2, q_2 be the initial and endpoints on $\partial\mathbb{D}$ of $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$, respectively (Figure 4). Then, the image trajectories $\tilde{\beta}_1 = \tilde{g}_h(\tilde{\alpha}_1)$ and $\tilde{\beta}_2 = \tilde{g}_h(\tilde{\alpha}_2)$ are well defined, according to §4.1: they have initial and endpoints $\tilde{g}(p_i)$ and $\tilde{g}(q_i)$, respectively, and subdivide their oriented slices in the proper ratio. (By projection, the image trajectories $\beta_i = g_h(\alpha_i)$, $i = 1, 2$, are determined, and so is the free homotopy class of arcs connecting β_1 and β_2 on S which corresponds to the class $[\gamma]$ on R.)

Lemma 5.3. *Let φ be simple. Choose arbitrary closed trajectories α_1 and α_2 of φ and an arc γ joining α_1 to α_2. Let $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ be a pair of lifts determined by γ. Then the corresponding trajectories $\tilde{\beta}_1$ and $\tilde{\beta}_2$ determined as above have the same vertical distance with respect to the image differential $\psi = g_h(\varphi)$: $\Delta(\tilde{\beta}_1, \tilde{\beta}_2) = \Delta(\tilde{\alpha}_1, \tilde{\alpha}_2)$.*

Proof. First, let $\alpha_1 = \alpha_2 = \alpha$ be the middle line of the cylinder of φ and let γ be an arc with initial and endpoint on α. For simplicitly, we can choose these to be the same point $P \in \alpha$. Let γ_0 be the geodesic with initial and endpoint P and homotopic to γ. Every slice, i.e., lift of the cylinder of φ, has two distinct endpoints. Unless it coincides with the slice containing $\tilde{\alpha}_1$ or $\tilde{\alpha}_2$ it either separates these two trajectories or it does not. Let N be the number of separating slices. Then

$$\Delta_\gamma(\alpha, \alpha) = \Delta(\tilde{\alpha}_1, \tilde{\alpha}_2) = (N + 1)b,$$

where b is the height of the slices of $\tilde{\varphi}$ (or the cylinder of φ).

Whether a slice separates $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ depends on whether its endpoints separate the endpoints of $\tilde{\alpha}_1$ from those of $\tilde{\alpha}_2$. The slices of $\tilde{\varphi}$ correspond to those of $\tilde{\psi}$ by the correspondence of their endpoints on $\partial\mathbb{D}$ which is given by $\tilde{g}$. Therefore the number of slices separating $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ is the same as the number of slices separating $\tilde{\beta}_1$ and $\tilde{\beta}_2$, showing that

$$\Delta(\tilde{\alpha}_1, \tilde{\alpha}_2) = \Delta(\tilde{\beta}_1, \tilde{\beta}_2).$$

In the general case, let α_1 and α_2 be arbitrary closed trajectories of φ and γ an arc joining them. The two lifts $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ are lying in well-defined slices. The vertical distance of the corresponding middle lines is invariant. We orient the cylinder of φ and take this orientation over to the cylinder of ψ by the homeomorphism g. By the lifting each slice is (individually) oriented, and this orientation is taken over by the boundary map $\tilde{g}$. Depending on the orientation we have to add or subtract vertical distances coming from the inside of the two slices. Since the topological situation is the same on both sides and the distances in the slices are the same too, we have shown the invariance of vertical distances in this case.

6. The mapping-by-heights for arbitrary holomorphic quadratic differentials

6.1. In this section we will frequently make use of the approximation by simple differentials on the surface R. It is useful to know that, unless the limit φ is itself simple, the heights of the cylinders of the approximating sequence tend to zero.

Lemma 6.1. *Let the sequence of normed simple differentials φ_n tend locally uniformly to φ. Assume that the heights of the cylinders of the φ_n are bounded away from zero. Then φ is simple and there exists n_0 such that $\varphi_n = \varphi$ for $n \geq n_0$.*

Proof. Denote the circumference of the cylinder A_n of φ_n by a_n, and its height by b_n. Then $a_n b_n = \|\varphi_n\| = 1$. Assume $b_n \geq b > 0$ for all n. It follows that the lengths a_n are bounded above, $a_n \leq 1/b$. Let α be a regular trajectory of φ and assume that it carries a nonclosed interval of length $c > 1/b$. Then α is the middle line of a φ-rectangle R_0 which is schlicht on R, and has length c and height $\varepsilon > 0$. For large enough n, there exists a regular φ_n-trajectory in R_0, since the φ_n approximate φ uniformly in R_0. Also, by the same reason, it must have φ_n-length greater than $1/b$. But all the regular trajectories of the φ_n are closed and have length $\leq 1/b$, which is a contradiction. Therefore, all the regular trajectories of φ are closed. There are only finitely many possible free homotopy classes. Each of these is equal to the class of α_n for all sufficiently large n. Therefore, φ is simple, with closed trajectories α, say. There exists n_0 such that $\alpha_n \sim \alpha$ for $n \geq n_0$. By the uniqueness for simple differentials with given homotopy of closed trajectories (see, e.g., [12]) we get $\varphi_n = \varphi$ for $n \geq n_0$.

6.2. We are now able to show that the mapping g_h, applied to an arbitrary holomorphic quadratic differential φ on a compact Riemann surface R, takes the regular trajectories of φ into those of $\psi = g_h(\varphi)$ and preserves the vertical distances. In the first step, using $\tilde{g}$ again, we show that the images of the endpoints of a regular trajectory of $\tilde{\varphi}$ are joined by a horizontal geodesic of $\tilde{\psi}$.

Lemma 6.2. *With the above notation let* p *and* q *be the endpoints of a regular trajectory* $\tilde{\alpha}$ *of* $\tilde{\varphi}$. *Then* $r = \tilde{g}(p)$ *and* $s = \tilde{g}(q)$ *are the endpoints of a horizontal geodesic* $\tilde{\beta}$ *of* $\tilde{\psi}$.

Proof. Let $\varphi \not\equiv 0$ be holomorphic on R, and not simple. Then φ can be approximated by a sequence of simple differentials φ_n (see, e.g., [5, Corollary 6.5]). The lifts $\tilde{\varphi}_n$ tend to $\tilde{\varphi}$ locally uniformly. Let $\tilde{\alpha}$ be a regular trajectory of $\tilde{\varphi}$, $z \in \alpha$. Choose a sequence of points $z_n \to z$ such that the trajectory $\tilde{\alpha}_n$ of $\tilde{\varphi}_n$ through z_n is regular (it is the lift of a closed trajectory of φ_n). By §2.3, the trajectories $\tilde{\alpha}_n$ tend uniformly (in the Euclidean metric) to $\tilde{\alpha}$. In particular, the endpoints p_n and q_n of $\tilde{\alpha}_n$ tend to the endpoints p and q of $\tilde{\alpha}$.

The differential $\psi_n = g_h(\varphi_n)$ is simple. Its lift $\tilde{\psi}_n$ has a regular trajectory $\tilde{\beta}_n$ with endpoints $\tilde{g}(p_n) = r_n$ and $\tilde{g}(q_n) = s_n$. In order to fix the ideas, we can take $\tilde{\beta}_n$ to be the middle line of the slice of $\tilde{\psi}_n$ connecting r_n and s_n. Since $\tilde{g}$ is continuous and $p_n \to p$ and $q_n \to q$, we have $r_n \to r = \tilde{g}(p)$ and $s_n \to s = \tilde{g}(q)$.

The approximating sequence $\{\varphi_n\}$ has heights converging to the heights of φ [5, Proposition 2.3]. Then, by [5, §5], the simple differentials $\psi_n = g_h(\varphi_n)$ converge in norm to $\psi = g_h(\varphi)$. Therefore, the sequence of lifts $\{\tilde{\psi}_n\}$ converges locally uniformly to the lift $\tilde{\psi}$ of ψ.

By Theorem 1 of [7] the points r and s are connected by a $\tilde{\psi}$-geodesic $\tilde{\beta}$. It is uniquely determined except for the case when r and s are joined by a family of parallel $\tilde{\psi}$-trajectories, which of course cannot be excluded a priori (see [7, Theorem 1, (ii)]). Using Lemma 2.7 we conclude that there exists a subsequence $\{\tilde{\psi}_{n_i}\}$ such that the corresponding regular trajectories $\tilde{\beta}_{n_i}$ converge uniformly to some $\tilde{\beta}$ connecting r and s. From the locally uniform convergence of the sequence $\{\tilde{\psi}_n\}$ to $\tilde{\psi}$ it follows easily that the geodesic arc $\tilde{\beta}$ is horizontal. q.e.d.

If the $\tilde{\psi}$-geodesic between r and s is unique, it follows by well-known arguments that the convergence to $\tilde{\beta}$ holds for the original sequence $\{\tilde{\psi}_n\}$ and arbitrary $\tilde{\alpha}_n \to \tilde{\alpha}$. If not, all geodesic connections of r and s belong to a horizontal slice.

6.3. The next step is to show that $\tilde{\beta}$ is actually a trajectory. Because of the possible nonuniqueness we have to work with the above subsequence $\{\tilde{\psi}_{n_i}\}$, which we however denote by $\{\tilde{\psi}_n\}$ again. Assume $\tilde{\beta}$ is not a trajectory. Then it is a subarc of a component $\tilde{\Gamma}$ of the critical horizontal graph of $\tilde{\psi}$. This component $\tilde{\Gamma}$ has at least one branching, possibly on only one side of $\tilde{\beta}$ (see Figure 5).

We choose a trajectory $\tilde{\alpha}'$ of $\tilde{\varphi}$ close to $\tilde{\alpha}$, with endpoints p' and q'. We approximate $\tilde{\alpha}'$ by trajectories $\tilde{\alpha}'_{n_i}$ of a subsequence $\{\tilde{\varphi}_{n_i}\}$ such that the trajectories $\tilde{\beta}'_{n_i}$ of the differentials $\tilde{\psi}'_{n_i}$ converge to a horizontal geodesic $\tilde{\beta}'$ connecting $r' = \tilde{g}(p')$ and $s' = \tilde{g}(q')$. Because $\tilde{g}$ is continuous, it is possible to choose $\tilde{\alpha}'$ such that $\tilde{\beta}'$ cuts $\tilde{\Gamma}$. By the invariance of vertical distances for simple differentials (Lemma 5.3) and the convergence of vertical distances (Lemma 4.1) we conclude that

$$\Delta(\tilde{\alpha}, \tilde{\alpha}') = \Delta(\tilde{\beta}, \tilde{\beta}') = \Delta(\tilde{\Gamma}, \tilde{\beta}') = 0,$$

which is a contradiction because $\tilde{\alpha}$ and $\tilde{\alpha}'$ necessarily have positive vertical distance. We have

Lemma 6.3. *Let p and $q \in \partial\mathbb{D}$ be the endpoints of a trajectory $\tilde{\alpha}$ of $\tilde{\varphi}$. Then $r = \tilde{g}(p)$ and $s = \tilde{g}(q)$ are the endpoints of a trajectory $\tilde{\beta}$ of $\tilde{\psi}$, $\psi = g_h(\varphi)$. If the geodesic connecting r and s is unique, then, for every sequence of simple differentials $\varphi_n \to \varphi$ and trajectories $\tilde{\alpha}_n \to \tilde{\alpha}$, the trajectories $\tilde{\beta}_n$ of the $\tilde{\psi}_n$ tend to $\tilde{\beta}$.*

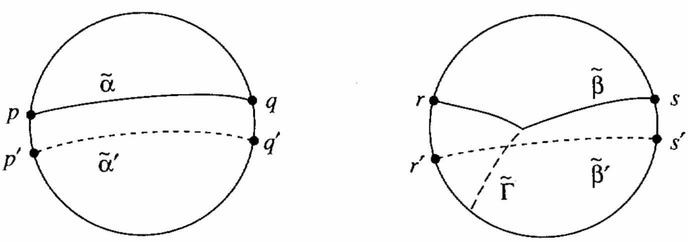

FIGURE 5

If, moreover, $\tilde{\beta}$ *and* $\tilde{\beta}'$ *are two unique trajectories corresponding to* $\tilde{\alpha}$ *and* $\tilde{\alpha}'$, *then they have the same vertical distance in the* $\tilde{\varphi}$*- and* $\tilde{\psi}$*-metric, respectively,*

$$\Delta_{\tilde{\varphi}}(\tilde{\alpha}, \tilde{\alpha}') = \Delta_{\tilde{\psi}}(\tilde{\beta}, \tilde{\beta}').$$

6.4. Let us now assume that the geodesic connection of r and s is not unique. Then there exists a (horizontal) slice $\widetilde{B}$ between r and s projecting onto a ring domain B swept out by closed trajectories of ψ.

Choose two trajectories $\tilde{\beta}$ and $\tilde{\beta}'$ in $\tilde{B}$. Going backwards by an analogous approximation procedure, but starting with a sequence $\{\psi_n\}$, we find that there are two trajectories $\tilde{\alpha}$ and $\tilde{\alpha}'$ connecting p and q with the same vertical distance. Therefore p and q are connected by a slice $\widetilde{A}$ which projects onto a ring domain A swept out by closed trajectories of φ. It follows from the above that both cylinders A and B must have the same height.

Orient $\widetilde{A}$ from p to q. Let $\tilde{\alpha}$ be the left boundary of $\widetilde{A}$. It is a right extreme horizontal geodesic connecting p and q (as such it is unique). Let $\tilde{\alpha}'$ be an arbitrary trajectory of $\widetilde{A}$. Approximating $\tilde{\alpha}$ from the right by trajectories $\tilde{\alpha}_n$ and $\tilde{\alpha}'$ arbitrarily by trajectories $\tilde{\alpha}_n'$ we see that the trajectories $\tilde{\beta}_n$ tend to the left boundary $\tilde{\beta}$ of $\widetilde{B}$, whereas the $\tilde{\beta}_n'$ tend to the trajectory $\tilde{\beta}'$. By Lemmas 5.3 and 4.1 we get

$$\Delta_{\tilde{\psi}}(\tilde{\beta}, \tilde{\beta}') = \Delta_{\tilde{\varphi}}(\tilde{\alpha}, \tilde{\alpha}').$$

Note that, for any convergent sequence of simple differentials with a limit which is not itself simple, the heights of the cylinders tend to zero. It is therefore not relevant which closed trajectory of the simple differential we concentrate on: e.g., we can always take the middle line.

Theorem 6.4. *Let* R *and* S *be compact Riemann surfaces of genus at least two, and let* $g\colon R \to S$ *be an orientation preserving homeomorphism. Then the mapping-by-heights* $g_h\colon \varphi \to \psi$ *takes the regular horizontal trajectories of* φ *onto those of* ψ. *The correspondence* $g_h\colon \alpha \to \beta$ *is determined by projection from the universal covering surfaces. The closed trajectories of a cylinder of* φ *are mapped onto the closed trajectories of the corresponding cylinder of* ψ.

In $\mathbb{D}$ we have the mapping $\tilde{g}_h\colon \tilde{\varphi} \to \tilde{\psi}$ of the lifts. The mapping of the trajectories $\tilde{g}_h\colon \tilde{\alpha} \to \tilde{\beta} = \tilde{g}_h(\tilde{\alpha})$ is given by the endpoints on $\partial\mathbb{D}$ and the mapping $\tilde{g}\colon \partial\mathbb{D} \to \partial\mathbb{D}$. Vertical distances are left invariant.

Note that the mapping does not extend to critical trajectories and critical points!

7. Construction of the Teichmüller mapping

7.1. Let $g: R \to S$ be an orientation preserving homeomorphism and $g_{\#}: Q_0(R) \to Q_0(S)$ the induced normalized mapping-by-heights. Assume a certain $\varphi \in Q_0(R)$ satisfies the equation $g_{\#}(-\varphi) = -g_{\#}(\varphi)$. Let $\psi = g_{\#}(\varphi)$. By Theorem 6.4, g_h and thus also $g_{\#}$ induces a mapping of the regular horizontal trajectories of φ onto those of ψ. Since the vertical trajectories of φ are the horizontal trajectories of $-\varphi$, it follows from the assumption that $g_{\#}$ also induces a mapping of the regular vertical trajectories of φ onto those of ψ. At the same time, the horizontal and vertical distances are multiplied by certain fixed factors λ and μ, respectively. These facts allow us to construct the Teichmüller mapping f homotopic to g, first on the universal covering surfaces, then by projection on the surfaces themselves.

As usual, we denote the lifts of φ and ψ to $\mathbb{D}_R$ and $\mathbb{D}_S$ by $\tilde{\varphi}$ and $\tilde{\psi}$, respectively. Besides, we fix a lift of g and denote the induced boundary mapping by $\tilde{g}: \partial\mathbb{D}_R \to \partial\mathbb{D}_S$. For any point $z \in \mathbb{D}_R$ which is the intersection of a regular horizontal and a regular vertical trajectory of $\tilde{\varphi}$ we define $w = \tilde{f}(z) \in \mathbb{D}_S$ to be the intersection of the image trajectories. This is well defined by the endpoints of these trajectories if none of them has a projection which is closed (belongs to a slice). But it is also well defined in the case of closed trajectories: recall that corresponding closed trajectories bisect their respective annuli (and slices in $\mathbb{D}$) proportionally, the factors being λ for vertical trajectories and μ for horizontal trajectories.

The mapping $\tilde{f}$ is thus defined on the set of intersections z of regular horizontal and vertical trajectories of $\tilde{\varphi}$ in $\mathbb{D}_R$. It is a bijection of this set onto the analogous set of $\tilde{\psi}$ in $\mathbb{D}_S$. The set is dense on every (regular) horizontal and vertical trajectory and f can therefore be extended by continuity to all of these entire lines.

Any one of the remaining points $z \in \mathbb{D}_R$ except for the zeros of $\tilde{\varphi}$ can be enclosed in an arbitrarily small rectangle, the sides of which are subintervals of horizontal and vertical trajectories. Letting the rectangles shrink to the point z we define $w = \tilde{f}(z)$ to be the intersection of the images of these rectangles. We see that $\tilde{f}$ is a homeomorphism of the complement of the zeros of $\tilde{\varphi}$ in $\mathbb{D}_R$ onto the complement of the zeros of $\tilde{\psi}$ in $\mathbb{D}_S$.

7.2. The next step is to show that $\tilde{f}$ is quasiconformal. It is in fact a Teichmüller map associated with $\tilde{\varphi}$ on $\mathbb{D}_R$ and $\tilde{\psi}$ on $\mathbb{D}_S$. To this end we again enclose an arbitrary point $z_0 \in \mathbb{D}_R$, $\tilde{\varphi}(z_0) \neq 0$, in a small φ-rectangle U and its image $w_0 = \tilde{f}(z_0)$ in the image rectangle U'. The maps

$$\zeta = \Phi(z) = \int^z \sqrt{\varphi(z)}\,dz \quad \text{and} \quad \zeta' = \Psi(w) = \int^w \sqrt{\psi(w)}\,dw$$

send U and U' conformally onto Euclidean rectangles. From the fact that horizontal distances of regular vertical trajectories are multiplied by a factor λ, whereas vertical distances of regular horizontal trajectories are multiplied by a factor μ (the φ- and ψ-distances are of course the Euclidean distances in the Φ- and Ψ-planes, respectively), and the continuity we conclude that the induced mapping from U onto U' is affine,

$$\zeta' = \xi' + i\eta' = \lambda\xi + i\mu\eta.$$

Therefore, $\tilde{f}$ is a Teichmüller mapping with dilatation $K = \max\{\frac{\lambda}{\mu}, \frac{\mu}{\lambda}\}$ associated with $\tilde{\varphi}$ and $\tilde{\psi}$. By continuity it extends to the zeros of $\tilde{\varphi}$. Furthermore, by construction, $\tilde{f}$ induces the same isomorphism between the universal covering groups as does $\tilde{g}$: $\tilde{f}T(\tilde{f})^{-1} = \tilde{g}T(\tilde{g})^{-1}$, where T is any cover transformation over R. Therefore, $\tilde{f}$ projects to the Teichmüller map $f: R \to S$ which is homotopic to g. We have

Theorem 7.2. *Let R and S be compact Riemann surfaces of genus ≥ 2 and let $g: R \to S$ be an orientation preserving homeomorphism. Let $g_\#: Q_0(R) \to Q_0(S)$ be the induced normalized mapping-by-heights. If there exists a $\varphi \in Q_0(R)$ such that $g_\#(-\varphi) = -g_\#(\varphi)$, then the differentials φ and $\psi = g_\#(\varphi)$ are Teichmüller differentials associated with g.*

8. Compact surfaces with punctures

8.1. Let R and S be compact Riemann surfaces, and let $\{X_\nu\}$ and $\{Y_\nu\}$ be finite sets of punctures on R and S, respectively. Let g be an orientation preserving homeomorphism of R onto S which takes the punctures X_ν onto the punctures Y_ν. Assume that the punctured surfaces $\dot{R} = R\backslash\{X_\nu\}$ and $\dot{S} = S\backslash\{Y_\nu\}$ have the universal covering surface $\mathbb{D}$.

We consider holomorphic quadratic differentials of finite norm on $\dot{R}$ and $\dot{S}$, respectively. They have at most first-order poles at the punctures. These points are added to the critical points of the differentials, even if these are reglar there and different from zero.

We will use approximation by simple differentials (see [5, Corollary 6.5]) and the Heights Theorem [5]. This allows the construction of the mapping-by-heights g_h associated with g, and of course the "normed mapping-by-heights" $g_\#$. We want to show that g_h has the same properties as in the compact case.

8.2. The definition of the vertical distance of two trajectories α_1 and α_2 with respect to a given free homotopy class of connecting arcs γ is the same as in §2.1. It is also true that we can fix two points $P_1 \in \alpha_1$ and $P_2 \in \alpha_2$ and only consider homotopic arcs connecting these two points. Homotopy is always meant on the punctured surface, where all arcs are supposed to lie. However, a shortest connection of P_1 and P_2 does in general not exist in $\dot{R}$: it will be a limiting arc γ_0 passing through some of the punctures X_ν. It consists of φ-straight arcs connecting critical points of φ (including some of the punctures). As such, it is uniquely determined (see [8]). In general, there is no curve in $\dot{R}$ connecting P_1 and P_2 in the given homotopy class which has minimal height. However, the limiting arc γ_0 has minimal height. The proof can be based on approximation by compact surfaces as in [8], using the fact that for compact surfaces the shortest curve also gives the smallest height (Lemma 3.2).

8.3. Convergence of vertical distances. Let the sequence $\{\varphi_n\}$ tend to φ in norm, $\|\varphi_n - \varphi\| \to 0$. This is equivalent to $\varphi_n \to \varphi$ locally uniformly on R, if we take square roots as local parameters near the first-order poles.

Let P, $P' \in R$ and let γ be a shortest curve (in the above sense) in the φ-metric joining P and P'. Let $P_n \to P$ and $P'_n \to P'$, and let γ_n be the shortest connection of P_n and P'_n in the φ_n-metric. According to §8.2 it is clear what we mean by saying that γ_n is in the same homotopy class as γ.

Then, because of the uniqueness of γ, the curves γ_n tend to γ, and so do their heights. We have

Lemma 8.3. *Let $\varphi_n \to \varphi$. Then the vertical distance of φ_n-trajectories in the φ_n-metric tends to the vertical distance of their limits with respect to the φ-metric.*

8.4. Let φ be holomorphic and of finite norm on an arbitrary Riemann surface. In [6] it was shown that geodesics on the universal covering surface $\mathbb{D}$ have well-determined, distinct endpoints. However, the asymptotic convergence of trajectories in $\mathbb{D}$ was only proved for compact underlying surfaces in [7], whereas we need it here for compact surfaces with punctures.

The boundary of the universal covering surface of a punctured surface is not at an infinite distance from the points of $\mathbb{D}$; every puncture can be reached along a critical trajectory ray of finite length, the lift of which is a boundary ray of $\mathbb{D}$ of finite length.

It is however not hard to see that every regular trajectory ray of φ (and therefore also its lift) has infinite length. If it is closed, this is evident. If it is not closed, it is recurrent and hence passes infinitely often through a fixed φ-rectangle of positive length on R, thus has infinite length.

With this property we can prove as in [7]

Lemma 8.4. *Let R, φ, and φ_n be as in* §8.1. *Assume that the sequence $\{\varphi_n\}$ converges in norm to φ. Let $\tilde{\alpha}$ be a regular trajectory of $\tilde{\varphi}$, $z \in \tilde{\alpha}$. Then, for every sequence of points $z_n \to z$ such that the trajectory $\tilde{\alpha}_n$ of $\tilde{\varphi}_n$ through z_n is regular, we have $\tilde{\alpha}_n \to \tilde{\alpha}$ uniformly in the Euclidean metric of the disk $\mathbb{D}$.*

8.5. We are now ready to prove the preliminary form of our main theorem.

Lemma 8.5. *Let p and q be the endpoints of a regular trajectory $\tilde{\alpha}$ of $\tilde{\varphi}$. Then $r = \tilde{g}(p)$ and $s = \tilde{g}(q)$ are the endpoints of a horizontal geodesic of $\tilde{\psi}$, where $\psi = g_h(\varphi)$.*

Proof. Let the sequence $\{\varphi_n\}$ of simple differentials approximate φ. Then there is a sequence of lifts $\tilde{\alpha}_n$ of the middle lines α_n of the cylinders of φ_n which approximate $\tilde{\alpha}$. Therefore, their endpoints p_n and q_n tend to p and q, respectively. The images $r_n = \tilde{g}(p_n)$ and $s_n = \tilde{g}(q_n)$ are the endpoints of properly chosen lifts of the middle lines of the cylinders of $\psi_n = g_h(\varphi_n)$. They tend to r and s, respectively, which are therefore the endpoints of a horizontal geodesic of $\tilde{\psi}$. q.e.d.

However, a horizontal geodesic β can go not only through zeros of ψ but just as well through punctures. As punctures correspond to parabolic fixed points, a lift $\tilde{\beta}$ of β can go through such fixed points, perhaps infinitely often. A successive pair of fixed points on $\tilde{\beta}$ bounds a segment of $\tilde{\beta}$ in $\mathbb{D}$, whose ends approach $\partial\mathbb{D}$ nontangentially.

8.6. In order to rule out both possibilities we first show that for arbitrary regular trajectories $\tilde{\alpha}$ and $\tilde{\alpha}'$ of $\tilde{\varphi}$ the image geodesics $\tilde{\beta}$ and $\tilde{\beta}'$ have the same vertical distance. This follows, as before, by approximation with simple differentials, for which the invariance of the vertical distance is already shown.

Assume now that the horizontal geodesic $\beta = g_h(\alpha)$ goes through a puncture of S. Then its lift $\tilde{\beta}$ has, besides its two endpoints r and s, another point, t, say, on $\partial\mathbb{D}$ (Figure 6). Approximating $\tilde{\alpha}$ on the vulnerable side by another regular trajectory $\tilde{\alpha}'$ it is easy to see that its image $\tilde{\beta}'$ must eventually meet $\tilde{\beta}$. This is only possible at a zero or a puncture (parabolic fixed point on $\partial\mathbb{D}$), as a look at Figure 6 shows. But horizontal arcs meeting at a puncture have the same height, because there are arbitrarily short arcs going around the puncture and joining them (even if the puncture represents a first-order pole of ψ). This contradicts the fact that $\tilde{\alpha}$ and $\tilde{\alpha}'$ have positive vertical distance. We have

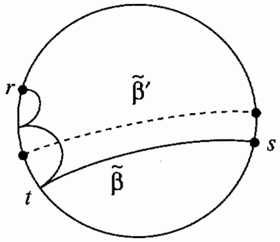

FIGURE 6

Theorem 8.6. *Let $\dot{R} = R\backslash\{X_\nu\}$ and $\dot{S} = S\backslash\{Y_\nu\}$ be punctured surfaces with the disk as universal covering surface. Let $g\colon R \to S$ be an orientation preserving homeomorphism with $g(X_\nu) = Y_\nu$ for all ν. Then the mapping-by-heights $\psi = g_h(\varphi)$, applied to the holomorphic quadratic differentials φ of finite norm on $\dot{R}$, takes the regular trajectories of φ into those of ψ and leaves their vertical distance invariant.*

8.7. A Teichmüller mapping $\tilde{f}$ with $\tilde{f}|\partial\mathbb{D} = \tilde{g}|\partial\mathbb{D}$ is constructed as before in the compact case. The lifted differentials $\tilde{\varphi}$ and $\tilde{\psi}$ have trajectory structures which are invariant under corresponding cover transformations T and $\theta(T)$, respectively. Therefore, $\tilde{f}$ is invariant. Since it induces the same isomorphism of the covering groups as the lift of g, its projection is homotopic to g.

We can now restate Theorem 7.2 for compact surfaces with punctures.

9. Application: The unique axis theorem

9.1. Fix a Riemann surface which arises from a compact surface of genus $g \geq 0$ from which $N \geq 0$ punctures have been removed, $3g+N-3>0$. We think now of Teichmüller space $T_{g,N}$ defined geometrically as the set of pairs $\{(S, f)\}$, where $f\colon R \to S$ is a quasiconformal map, subject to the equivalence $(S_1, f_1) \equiv (S, f)$ if $f_1 f^{-1}\colon S \to S_1$ is homotopic to a conformal map. It is a complete metric space in the Teichmüller metric $d(p, q) = \frac{1}{2}\log K(p, q)$, where $K(p, q)$ is the maximal dilatation of the Teichmüller map homotopic to $f_1 f^{-1}\colon S \to S_1$, with $p = (S, f)$ and $q = (S_1, f_1) \in T_{g,N}$. Of course, it is a complex analytic manifold of dimension $3g+N-3$, but here we will only use metric properties.

Let $h\colon R \to R$ be a quasiconformal automorphism. It determines the bijection $\tilde{h}$,

$$\tilde{h}\colon (S, f) \to (S, fh),$$

which is an isometry of $T_{g,N}$. Conversely, it is a result of Earle-Kra that every isometry of $T_{g,N}$ is of the form $\tilde{h}$. The group of all isometries is discrete and is called the Teichmüller modular group.

The Teichmüller line L through $(S, f) \in T_{g,N}$ in the direction φ, where φ is a quadratic differential of finite norm on S, is the set $\{(S_t, f_t f)\}$. Here $f_t: S \to S_t$ is the Teichmüller map satisfying

$$(f_t)_{\bar{z}}/(f_t)_z = t\overline{\varphi}/|\varphi|, \qquad -1 < t < 1.$$

If the isometry $\tilde{h}$ maps a line L onto itself, then L is called an *axis* of $\tilde{h}$, in line with the conventional terminology for hyperbolic Möbius transformations in the disk.

The Teichmüller line L from $(S, f) \in T_{g,N}$ through (S, fh) has direction φ, where φ is associated with the Teichmüller map h_0 homotopic to $fhf^{-1}: S \to S$. The line L is itself preserved by $\tilde{h}$ if and only if the point (S, fhh) also lies on L. A necessary and sufficient condition for this to happen is that h_0 be a stable (uniquely extremal) map: that $-\varphi$ be the differential associated with h_0^{-1} ([1], [9]).

An element $\tilde{h}$ of the Teichmüller modular group has an axis if and only if it arises from a pseudo-Anosov automorphism h (for expositions of pseudo-Anosov maps see, for example, [2] or [9]). Bers [1] showed that such elements have axes (see also [4] and [9]). Conversely, if $\tilde{h}$ preserves the line L through (S, f) in direction φ, then $fhf^{-1}: S \to S$ is homotopic to a stable Teichmüller map associated with φ which can only be pseudo-Anosov. The uniqueness of the axis L goes deeper, in that it requires extensive knowledge of the map $g_\#$.

9.2. Theorem 9.2. *A pseudo-Anosov map acting on $T_{g,N}$ has a unique axis.*

Remark. The only reference we have to an earlier treatment is Lemma 16 of Exposé 12 by Fathi and Póenaru in [2]. We are grateful to the referee for the remark that it also follows from Royden's theorem that the Teichmüller and the Kobayashi metrics coincide.

Proof of Theorem 9.2. We may choose the base point (R, id) to lie on an axis L of the pseudo-Anosov map $\tilde{h}$, and to have direction φ at (R, id). Hence, $h_\#(\varphi) = \varphi$ and $h_\#(-\varphi) = -\varphi$.

Suppose (S, f) lies on a possibly different axis in direction ψ. Then on S, $(fhf^{-1})_\#(\psi) = \psi$ and $(fhf^{-1})_\#(-\psi) = -\psi$. We claim that $f_\#(\varphi) = \psi$ and $f_\#(-\varphi) = -\psi$. This is true for the following reason. For any simple loop γ on R, the sequence of simple differentials $\{\varphi[h^n(\gamma)]\}$ converges to φ as $n \to +\infty$ and to $-\varphi$ as $n \to -\infty$ (see [9]). Likewise on S, $\{\varphi[fh^n f^{-1} f(\gamma)]\}$ converges to ψ as $n \to +\infty$ and to $-\psi$ as

$n \to -\infty$. Furthermore, $f_\#(\varphi[h^n(\gamma)]) = \varphi[fh^n(\gamma)]$. Our assertion is now a consequence of the continuity of $f_\#\colon Q_0(R) \to Q_0(S)$.

All that remains is to apply the construction of §6: the Teichmüller map homotopic to f is itself associated with φ. This means $(S, f) \in L$, and $\psi = \varphi$. The axis L_1 coincides with L.

10. A geometric proof of the Teichmüller mapping theorem

10.1. In the last part of this paper we give a new proof of the Teichmüller mapping theorem. It is based on the characterization of the Teichmüller differentials developed in the first part and consists in solving the relation

$$g_\#(-\varphi) = -g_\#(\varphi)$$

for φ. It is however easier to work with the horizontal stretching version, which is equivalent with the above.

Assume φ and ψ are the Teichmüller differentials associated with g. Then $\psi = g_h(\varphi)$, which means that the heights are the same. On the other hand, the horizontals are stretched by $K > 1$ (we assume that there is no conformal mapping homotopic to g). Therefore, $\|g_h(\varphi)\| = \|\psi\| = K\|\varphi\|$. In the next section we will establish a general norm inequality for the mapping g_h. The Teichmüller differentials will then turn out to be the solutions of an extremum problem related to this inequality.

10.2. Let R and R' be compact Riemann surfaces with distinguished points P_ν and P'_ν, respectively. Let $g\colon R \to R'$, $g(P_\nu) = P'_\nu$ $\forall\nu$, be a homeomorphism. The quadratic differentials φ on R and ψ on R' are holomorphic on the punctured surfaces $R\backslash\{P_\nu\}$ and $R'\backslash\{P'_\nu\}$, respectively, and have at most simple poles at the punctures.

Theorem 10.2. *Let g be a quasiconformal mapping with maximal dilatation K. Then, for any φ on R and $\psi = g_h(\varphi)$ on R', the norm inequality*

$$\frac{1}{K}\|\varphi\| \le \|\psi\| \le K\|\varphi\| \tag{1}$$

holds.

Proof. Assume first that φ has closed trajectories. Denote the ring domains swept out by the trajectories by R_i, their moduli by M_i, and their heights by b_i. Let $\widetilde{R}_i = g(R_i)$ have the modulus $\widetilde{M}_i$. Then

$$\frac{1}{\widetilde{M}_i} \le K\frac{1}{M_i}. \tag{2}$$

After multiplication with b_i^2 and summation over all ring domains we find

$$\Sigma \frac{b_i^2}{\widetilde{M}_i} \le K \Sigma \frac{b_i^2}{M_i}. \tag{3}$$

The sum on the left-hand side is minimized by the set of ring domains R_i' of the quadratic differential $\psi = g_h(\varphi)$ (for a proof of the minimum property see [12, Theorem 20.5]). Denote their moduli by M_i'; the heights are of course the same $b_i' = b_i$. Thus,

$$\Sigma \frac{b_i^2}{M_i'} \le \Sigma \frac{b_i^2}{\widetilde{M}_i} \le K \Sigma \frac{b_i^2}{M_i}. \tag{4}$$

The left- and right-hand side of (4) are the norms of ψ and φ, respectively. This proves the right-hand side of (1) for quadratic differentials with closed trajectories. To prove the left-hand side we start with ψ and go backwards to $\varphi = g_h^{-1}(\psi)$: we get $\|\varphi\| \le K\|\psi\|$. For the general case we use approximation by differentials with closed trajectories. q.e.d.

Of course one can replace K by its smallest value, i.e., the maximal dilatation K of an extremal quasiconformal mapping homotopic to g.

10.3. By Theorem 10.2, the quotients

$$\frac{\|\psi\|}{\|\varphi\|}, \frac{\|\varphi\|}{\|\psi\|}, \qquad \psi = g_h(\varphi), \tag{5}$$

are bounded. On the other hand, the mapping g_h clearly is homogeneous with respect to positive constants,

$$g_h(\lambda\varphi) = \lambda g_h(\varphi), \qquad \lambda > 0. \tag{6}$$

Therefore, one can normalize φ, $\|\varphi\| = 1$, in the quotients (5). It follows by the continuity of g_h and the compactness of the unit sphere $\{\varphi; \|\varphi\| = 1\}$ that they have a maximum value. Setting

$$L = \sup\left\{\frac{\|\psi\|}{\|\varphi\|}, \frac{\|\varphi\|}{\|\psi\|}\right\} \tag{7}$$

we may assume that L is attained by the quotient $\|\psi\|/\|\varphi\|$. We thus have

$$\|\psi\| = \|g_h(\varphi)\| = L\|\varphi\| \tag{8}$$

with maximal L, $1 \le L \le K$.

10.4. The proof of the Teichmüller mapping theorem consists in showing that the pair of differentials φ, $\psi = g_h(\varphi)$ of (8) satisfies the relation

$$g_h(-\varphi) = \text{const}.(-\psi), \tag{9}$$

with a positive constant (which will be $1/K^2$).

Theorem 10.4. *Let $\dot{R}$ and $\dot{R}'$ be compact surfaces with punctures, and let $g\colon \dot{R} \to \dot{R}'$ be a quasiconformal homeomorphism. Denote by g_h the induced mapping-by-heights of the holomorphic quadratic differentials with finite norm on $\dot{R}$ onto the analogous set on $\dot{R}'$. Assume that*

$$\|g_h(\varphi)\| = L\|\varphi\| \tag{10}$$

and that $L \geq 1$ is maximal. Then $g_h(-\varphi) = \text{const}.(-g_h(\varphi))$, with a positive constant.

Proof. In [13] the goal was achieved by a variational method established in [12]. This was possible because the quadratic differentials associated with the configurations all had closed trajectories. On a generic compact surface this is however not the case. But for differentials with one cylinder the variational lemma becomes superfluous. This is the reason for the following approximation procedures by simple differentials.

Let φ and $\psi = g_h(\varphi)$ satisfy (10), and let

$$\tilde{\varphi} = g_h^{-1}(-\psi). \tag{11}$$

(Note that in this section we use the tilde to denote a variation of the given quantity.) We want to show that $\tilde{\varphi} = \text{const}(-\varphi)$, with some positive constant. We approximate φ by a sequence of simple differentials φ_n (if φ itself is simple, we set $\varphi_n = \varphi$ for all n; the same in the following approximations). Let

$$\psi_n = g_h(\varphi_n). \tag{12}$$

We have $\psi_n \to \psi$ locally uniformly on $\dot{R}'$. The ring domains of the differentials φ_n are denoted by R_n, and those of ψ_n are denoted by R'_n. Of course we have the limit relations

$$-\psi_n \to -\psi, \tag{13}$$

and by continuity of the mapping-by-heights

$$\tilde{\varphi}_n := g_h^{-1}(-\psi_n) \to \tilde{\varphi} = g_h^{-1}(-\psi). \tag{14}$$

The differentials $-\psi_n$ are in general not simple (nor do they have closed trajectories). Therefore we again approximate $-\psi_n$ for every fixed n by a sequence of simple differentials

$$\psi_{nk} \to -\psi_n, \qquad k \to \infty. \tag{15}$$

We set

$$\tilde{\varphi}_{nk} = g_h^{-1}(\psi_{nk}) \tag{16}$$

and get

$$\tilde{\varphi}_{nk} \to \tilde{\varphi}_n, \qquad k \to \infty \text{ for each } n. \tag{17}$$

The ring domains of the ψ_{nk} are denoted by R'_{nk}, and their heights (in the ψ_{nk}-metric) are denoted by d'_{nk}. Correspondingly, the cylinders of the $\tilde{\varphi}_{nk}$ are R_{nk}, with heights $d_{nk} = d'_{nk}$. Note that the trajectories of the $-\psi_n$ are the vertical trajectories of the ψ_n. Therefore, the heights of the $-\psi_n$ are the horizontal lengths of the ψ_n. This is nothing but the equation

$$\int_\gamma |\operatorname{Im}\{\sqrt{-\psi_n}\,dz\}| = \int_\gamma |\operatorname{Re}\{\sqrt{\psi_n}\,dz\}|.$$

With this setting we can now compute the norm of $\tilde{\varphi}$. We denote a generic local parameter of $\dot{R}$ by $z = x + iy$, whereas the parameter determined by φ_n is

$$z_n = x_n + iy_n = \int^z \sqrt{\varphi_n(z)}\,dz. \tag{18}$$

We work in the ring domain R_n of φ_n, with the parameter z_n. Note that R_n covers the entire surface R up to finitely many analytic arcs. Let α_n be an arbitrary closed trajectory of φ_n in R_n. Its length in the $\tilde{\varphi}_{nk}$-metric, in terms of the parameter z_n and with the notation

$$d\tilde{w}_{nk} = d\tilde{u}_{nk} + i\,d\tilde{v}_{nk} = \sqrt{\tilde{\varphi}_{nk}(z_n)}\,dz_n, \tag{19}$$

is

$$\begin{aligned} \int_{\alpha_n} |d\tilde{w}_{nk}| &= \int_{\alpha_n} |\tilde{\varphi}_{nk}|^{1/2} |dz_n| \\ &= \int_{\alpha_n} \sqrt{(d\tilde{u}_{nk})^2 + (d\tilde{v}_{nk})^2} \ge \int_{\alpha_n} |d\tilde{v}_{nk}|. \end{aligned} \tag{20}$$

Clearly the last term is greater or equal to the height of α_n with respect to the differential $\tilde{\varphi}_{nk}$, which is the same as the height of α'_n with respect to ψ_{nk}, by definition of the differentials and because $\alpha'_n \sim g(\alpha_n)$. Therefore inequality (20) continues with

$$\int_{\alpha_n} |d\tilde{v}_{nk}| \ge h_{\tilde{\varphi}_{nk}}[\alpha_n] = h_{\psi_{nk}}[\alpha'_n]. \tag{21}$$

Now let $k \to \infty$ for fixed n and fixed α_n. Since $\psi_{nk} \to -\psi_n$, we have

$$h_{\psi_{nk}}[\alpha'_n] \to h_{\psi_n}[\alpha'_n] = a'_n = L_n a_n. \tag{22}$$

Here, a_n is the φ_n-length of α_n, a'_n is the ψ_n-length of α'_n, and L_n is defined by the last equation. From $\|\varphi_n\| = a_n b_n$, $\|\psi_n\| = a'_n b_n$, $\|\varphi_n\| \to$

$\|\varphi\|$, and $\|\psi_n\| \to \|\psi\| = L\|\varphi\|$, we conclude $L_n \to L$ for $n \to \infty$. Since $dz_n = dx_n + idy_n = dx_n$ along α_n, we have in the limit $k \to \infty$

$$\tag{23} \begin{aligned} \int_{\alpha_n} |\tilde{\varphi}_n(z_n)|^{1/2}\, dx_n &= \int_{\alpha_n} \sqrt{\left(\frac{\partial \tilde{u}_n}{\partial x_n}\right)^2 + \left(\frac{\partial \tilde{v}_n}{\partial x_n}\right)^2}\, dx_n \\ &\geq \int_{\alpha_n} \left|\frac{\partial \tilde{v}_n}{\partial x_n}\right| dx_n \geq L_n a_n. \end{aligned}$$

Integrating over y_n from zero to b_n we find

$$\tag{24} \begin{aligned} \int_{R_n}\!\!\int |\tilde{\varphi}_n(z_n)|^{1/2}\, dx_n\, dy_n &= \int_{R_n}\!\!\int \sqrt{\left(\frac{\partial \tilde{u}_n}{\partial x_n}\right)^2 + \left(\frac{\partial \tilde{v}_n}{\partial x_n}\right)^2}\, dx_n\, dy_n \\ &\geq \int_{R_n}\!\!\int \left|\frac{\partial \tilde{v}_n}{\partial x_n}\right| dx_n\, dy_n \\ &\geq L_n a_n b_n = L_n\|\varphi_n\|. \end{aligned}$$

Because the ring domain R_n covers R, this is in fact an integral over the entire surface. For a passage to the limit $n \to \infty$ we switch to fixed local parameters, which we generically call $z = x + iy$. The transformation rules give, with an evident slight abuse of notation,

$$\tag{25} |\tilde{\varphi}_n(z_n)|^{1/2} = |\tilde{\varphi}_n(z)|^{1/2} \left|\frac{dz}{dz_n}\right|,$$

$$\tag{26} dx_n\, dy_n = \left|\frac{dz_n}{dz}\right|^2 dx\, dy,$$

$$\tag{27} |dz_n| = |\varphi_n(z)|^{1/2} |dz|.$$

The inequality (24) without the middle terms now reads

$$\tag{28} \int_R\!\!\int |\tilde{\varphi}_n(z)|^{1/2} |\varphi_n(z)|^{1/2}\, dx\, dy \geq L_n\|\varphi_n\|.$$

Letting $n \to \infty$, this gives

$$\tag{29} \int_R\!\!\int |\tilde{\varphi}(z)|^{1/2} |\varphi(z)|^{1/2}\, dx\, dy \geq L\|\varphi\|.$$

Applying the Schwarz inequality and dividing by $\|\varphi\|$ we finally find

$$\tag{30} \|\tilde{\varphi}\| \geq L^2\|\varphi\| = L\|\psi\|.$$

Here, equality must hold, because L is maximal.

10.5. In order to see that in fact $\tilde{\varphi} = c\varphi$ with $c = L^2$, we specify the parameter z to $z = \int \sqrt{\varphi}$, which is legitimate on $\dot{R}$ outside the zeros of φ. In terms of this parameter we have $\varphi \equiv 1$. Therefore (24), in the limit $n \to \infty$, becomes (we have equality in (29)!)

$$(32)\qquad \begin{aligned} L\|\varphi\| &= \int_R\int |\tilde\varphi(z)|^{1/2}\,dx\,dy = \int_R\int \sqrt{\left(\frac{\partial \tilde u}{\partial x}\right)^2 + \left(\frac{\partial \tilde v}{\partial x}\right)^2}\,dx\,dy \\ &\geq \int_R\int \left|\frac{\partial \tilde v}{\partial x}\right|\,dx\,dy = L\|\varphi\|. \end{aligned}$$

We conclude that $\partial\tilde u/\partial x \equiv 0$, which means that the trajectories of $\tilde\varphi$ are orthogonal to the trajectories of φ, and hence

$$(33)\qquad \tilde\varphi = g_h^{-1}(-\psi) = -L^2\varphi.$$

Applying g_h, we get

$$(34)\qquad g_h(-\varphi) = \frac{1}{L^2}(-\psi).$$

Of course, L is the dilatation of the Teichmüller mapping f homotopic to g and constructed in §7. We put again $L = K$.

Corollary. *Equality on the right-hand side of* (1) *holds iff* φ *and* $\psi = g_h(\varphi)$ *are the Teichmüller differentials associated with* g, *and the Teichmüller mapping* f *is the horizontal stretching by* K; *equality on the left iff* φ *and* ψ *are the Teichmüller differentials and* f *is the horizontal contraction by* $1/K$.

Proof. Assume φ and $\psi = g_h(\varphi)$ are the Teichmüller differentials, and f is the horizontal stretching by K. Then, evidently $\|\psi\| = K\|\varphi\|$ (see §1.1). Conversely, let $\|\psi\| = K\|\varphi\|$ and $\psi = g_h(\varphi)$. By (1), the factor K is maximal, therefore φ and ψ are the Teichmüller differentials associated with g. Since K is the smallest maximal dilatation for the mappings homotopic to g, the Teichmüller mapping is the horizontal stretching by K.

To prove the second statement, we look at the inverse mapping $\varphi = g_h^{-1}(\psi)$.

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